The Lee Fields Medal — SOLUTIONS

1. Using each number exactly once, place the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 in a 3×3 square so that the rows, columns and diagonals sum to the same total.

   Solution: The numbers have a sum of

   $$(1 + 9) + (2 + 8) + (3 + 7) + (4 + 6) + 5 = 45.$$

   Therefore each row-, column-, and diagonal-sum must be 15. The number placed in the middle must be in every sum and therefore every number will be combined with the middle number to sum to 15. We cannot place a number strictly less than five in the middle because in each case, the smallest number remaining added to this middle number gives a maximum of five $(4 + 1)$ and the largest this sum can be is found by adding nine to give 14. Similarly, we cannot place a number strictly greater than five in the middle because, in each case, the largest number remaining added to this middle number gives a minimum of 15 $(6+9)$, and adding anything to this gives something strictly larger than five. Therefore, if the task is possible at all, five must be placed in the middle.

   Now if we place one in a corner, say the top left, then we must have nine in the opposite corner. Eight must avoid nine, say by going in row 2, column 1. This means two must go across from eight and then four above two: but then the first row sum cannot be 15.

1	!	4
8	5	2
6		9

   Therefore one cannot go in a corner. If we try two in a corner we are led to a solution. Eight must be opposite two, nine must avoid eight, one must go opposite nine, six must go above one, seven must go between two and six, three below five, four opposite six and by inspection this is a solution.

2	7	6
9	5	1
4	3	8

   Note that there are four choices for two, then two choices for nine, and after that there are no more choices. If two does not go in a corner one ends up with a row or column with a nine and a three and 15 cannot be made without a second three. Therefore there are eight possible solutions.

   Remark: Any of the eight solutions confers full marks.

2. Professor Oldie does not believe in calculators. You have to prove it to him on paper, using mathematical considerations, that

   $$\sqrt{10} > \sqrt{2} + \sqrt{3}.$$

   You may not use approximations nor your calculator.

   Solution: Let P be the proposition that $\sqrt{10} > \sqrt{2} + \sqrt{3}$. In roughwork, it can be seen that if P is true, then $25 > 24$, which is true. Running the roughwork backwards:

   $$\begin{aligned} & 25 > 24 \\ \underset{\surd}{\Rightarrow}\ & 5 > \sqrt{24} = 2\sqrt{6} \\ \underset{+5}{\Rightarrow}\ & 10 > 2\sqrt{2}\sqrt{3} + 5 \\ \Rightarrow\ & 10 > 2 + 2\sqrt{2}\sqrt{3} + 3 \\ \Rightarrow\ & 10 > (\sqrt{2} + \sqrt{3})^2 \\ \underset{\surd}{\Rightarrow}\ & \sqrt{10} > \sqrt{2} + \sqrt{3} \qquad \bullet \end{aligned}$$

Remark: Full marks should not be awarded for

$$P \Rightarrow 25 > 24 \Rightarrow P.$$

This comprises circular reasoning, and such logic can fail. For example, let T be the statement that

$$-2 > 1.$$

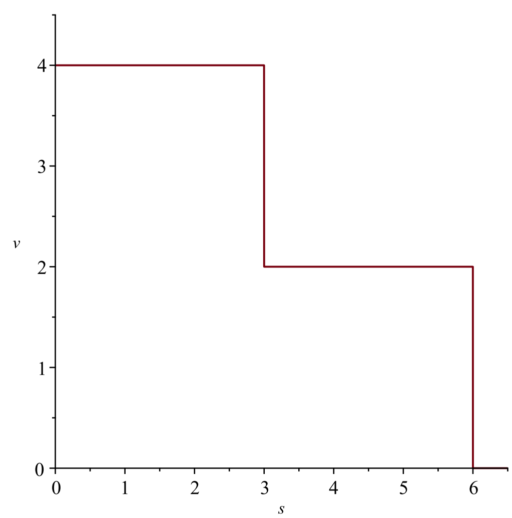
$$T \Rightarrow 4 > 1,$$

which is true, but this does not imply that T is true.

3. CIT students Rebecca and Aoife share an apartment which is 6 km away from college. One day they left the apartment at the same time but decided to walk to the college separately. Rebecca walked first half of the *distance* with the speed of 4 km/h and the second half of the *distance* at 2 km/h. Aoife had a different plan. She walked the first half of the *time* with the speed of 4 km/h and walked the remaining half of the *time* at 2 km/h. Who got to college first? Show all calculations.

   Algebra Solution:

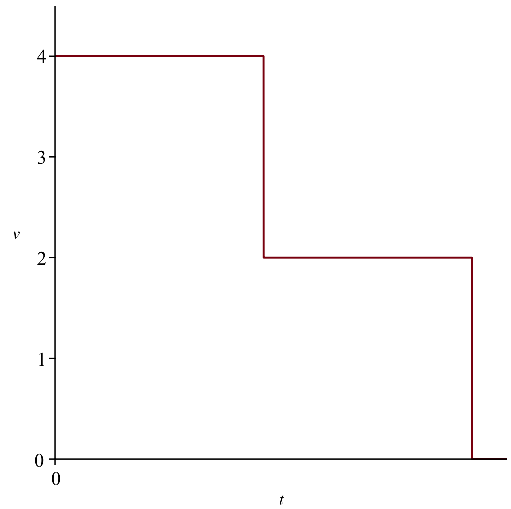
   Consider Rebecca first. Her situation can be captured by a graph of speed vs distance:



   Let the first time interval be of length t_1, and the second t_2: Using $s = vt \Rightarrow t = s/v$:

$$\begin{aligned} t_{\text{Rebecca}} &= t_1 + t_2 \\ &= \frac{3}{4} + \frac{3}{2} = \frac{9}{4} \text{ hours.} \end{aligned}$$

Now consider Aoife. Her situation can be captured by a graph of speed vs time:



Both time intervals are equal, to say t. We also know the two distances she travels sum to six. Using $s = vt$:

$$\begin{aligned} s_1 + s_2 &= 6 \\ \Rightarrow 4t + 2t &= 6 \\ \Rightarrow 6t &= 6 \\ \Rightarrow t &= 1 \\ \Rightarrow t_{\text{Aoife}} &= 2 \text{ hours}, \end{aligned}$$

therefore Aoife got to college 15 minutes earlier than Rebecca.

The 'Why' Solution: Imagine Rebecca and Aoife walking together. From the above calculation we know that Aoife will walk for one hour at 4 km/h and one hour at 2 km/h. This means that in the first hour Aoife is due to walk 4 km. Before this happens though, when they reach the 3 km mark Rebecca stops and say that I am now halfway and so going to walk at 2 km/h the rest of the way. This happens after 45 minutes. Aoife however carries on walking at 4 km/h for 15 minutes while Rebecca is walking at 2km/h. In this 15 minutes Aoife is picking up 2 km/h on her housemate which over 15 minutes translates to a 500 m lead by the time she reaches the 4 km mark. At this point, both Rebecca and Aoife are travelling at the same speed but Aoife is 500 m ahead. When Aoife arrives at college Rebecca still has 500 m to walk, which at 2 km/h translates to 15 minutes.

4. Explain geometrically why the set of simultaneous equations

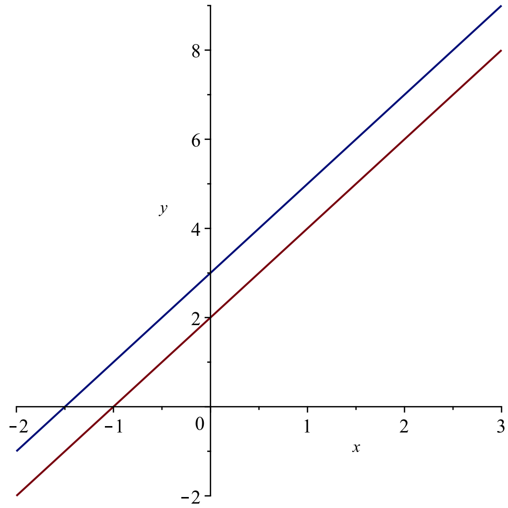
$$\begin{aligned} 2x - y &= -3 \\ 2x - y &= -2 \end{aligned}$$

has *no* solution.

Solution: Rewrite both equations in the form $y = mx + c$:

$$\begin{aligned} y &= 2x + 3 \\ y &= 2x + 2 \end{aligned}$$

These are lines, with equal slopes ($m = 2$) but different y-intercepts two and three:



As the lines are parallel, there is no point (x_0, y_0) that is on both lines. Therefore no pair of numbers (x_0, y_0) that satisfy the equations simultaneously.

5. Which is a closer fit, a square peg in a round hole, or a round peg in a square hole? Justify your answer fully.

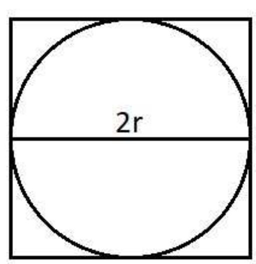
   Solutions: There are two issues to consider before calculations are made.

   The first is to define how good a fit is. There are a number of different ways of doing this, but all reasonable ways should give the same answer. What we might run with here is that the closeness of fit of a shape $\mathcal{S}_1$ into $\mathcal{S}_2$ is given by

$$\text{goodness of fit, GoF} := \frac{A(\mathcal{S}_1)}{A(\mathcal{S}_2)}.$$

   The second thing to query is whether the question is well defined. Perhaps for small pegs, square into round is better; and for larger pegs, round into square is better? Picking a specific radius and sidelength — for this very reason — does not answer the question "fully". To remove this difficulty we must work in the abstract: if we choose the round peg to have radius r — and the GoF is independent of r — and similarly for the square plug if the GoF does not depend on the sidelength s — then the question is well defined.

   First consider a round peg of radius r:

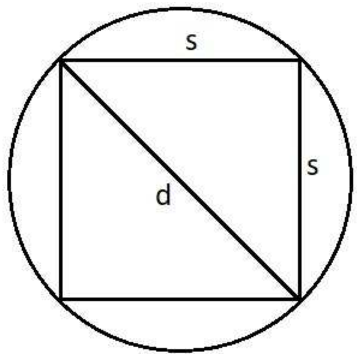


We have $A(\circ_1) = \pi r^2$ and $A(\Box_1) = 4r^2$,

$$\text{GoF(round peg)} = \frac{A(\circ_1)}{A(\Box_1)} = \frac{\pi r^2}{4r^2} = \frac{\pi}{4},$$

which is independent of r (✓).

Now consider a square peg of sidelength s:



The widest part of a square is the diagonal and so the diagonal is the diameter of the round hole. Using Pythagoras Theorem we have

$$d^2 = s^2 + s^2 \Rightarrow d = \sqrt{2}s \Rightarrow r = \frac{1}{\sqrt{2}}s,$$

the radius of the round peg. We have $A(\circ_2) = \frac{\pi}{2}s^2$ and $A(\Box_2) = s^2$,

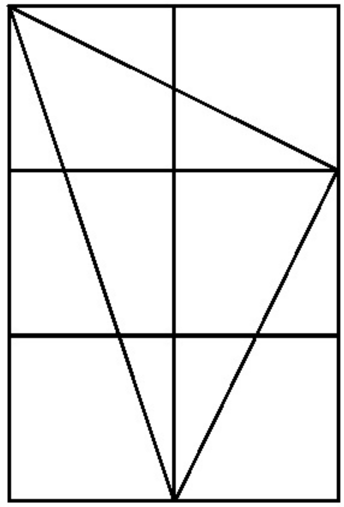
$$\text{GoF(square peg)} = \frac{A(\Box_2)}{A(\circ_2)} = \frac{s^2}{\frac{\pi}{2}s^2} = \frac{2}{\pi},$$

independent of s. We can of course go to the calculator to see that GoF(round peg) $\approx 78.54\%$ is better than GoF(square peg) $\approx 63.66\%$. Alternatively, as $\pi > 3$

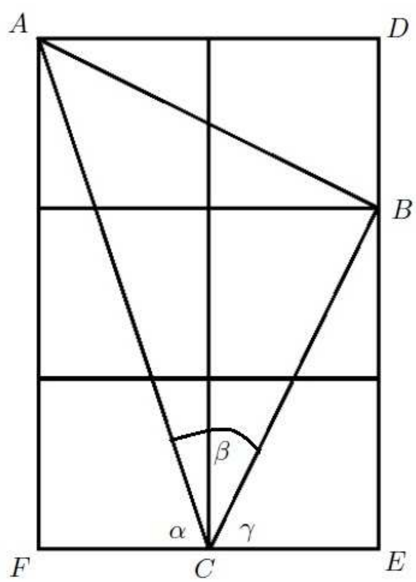
$$\begin{aligned} \pi^2 &> 8 \\ \Rightarrow \pi \cdot \pi &> 2 \times 4 \\ \underset{\div 4\pi}{\Rightarrow} \frac{\pi}{4} &> \frac{2}{\pi} \\ \Rightarrow \text{GoF(round peg)} &> \text{GoF(square peg)} \end{aligned}$$

6. Use the below figure to prove that

$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi = 180°.$$



Solution: In contrast to the previous question where the independence of the results from the variables s and r was important, we can just choose the squares here to be of length one. Denote the following points and angles:



It is a square grid so we have already that

$$\tan\alpha = \frac{|AF|}{|CF|} = 3 \Rightarrow \alpha = \tan^{-1}(3),$$
$$\tan\gamma = \frac{|BE|}{|CE|} = 2 \Rightarrow \gamma = \tan^{-1}(2).$$

We apply Pythagoras Theorem to ΔABD, ΔBCE, and ΔACF to get

$$|AB|^2 = |AD|^2 + |BD|^2 = 5 \Rightarrow |AB| = \sqrt{5},$$
$$|BC|^2 = |BE|^2 + |CE|^2 = 5 \Rightarrow |BC| = \sqrt{5},$$
$$|AC|^2 = |AF|^2 + |CF|^2 = 10 \Rightarrow |AC| = \sqrt{10}.$$

Note that

$$|AC|^2 = |AB|^2 + |BC|^2,$$

so by the converse to the Pythagoras Theorem ΔABC is a right-angled triangle and so we can calculate

$$\tan\beta = \frac{|AB|}{|BC|} = \frac{\sqrt{5}}{\sqrt{5}} = 1 \Rightarrow \beta = \tan^{-1}(1).$$

The angle $\alpha+\beta+\gamma$ is equal to a straight angle, $180° = \pi$, but also equal to $\beta+\gamma+\alpha$,

$$\Rightarrow \tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi \qquad \bullet$$

7. If you have 23 people in a room, what is the probability that at least two of them share a birthday?

   Solution: We make the assumption here that the birthdays are independently and identically distributed uniformly amongst the 365 non-Leap Days.

   Let A be the event that the 23 people $\{P_1, P_2, \dots, P_{23}\}$ have different birthdays. The event A is the same as

$$(P_2 \text{ has a different birthday to } P_1) \cap (P_3 \text{ has a different birthday to } P_2 \text{ and } P_1) \cap \cdots$$
$$\cap (P_{23} \text{ has a different birthday to } P_1 \text{ and } P_2 \text{ and ... and } P_{22}).$$

   With the assumption of independence ($\mathbb{P}[E_1 \cap E_2] = \mathbb{P}[E_1] \cdot \mathbb{P}[E_2]$) we have

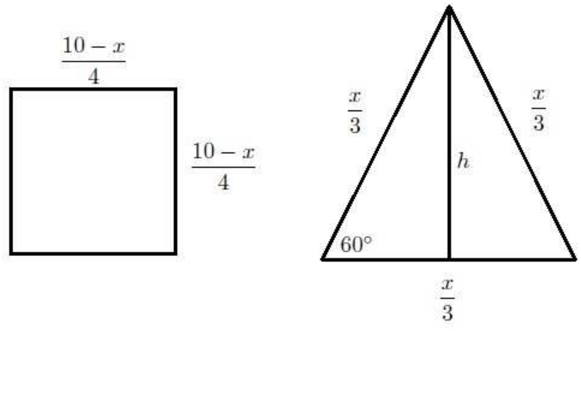
$$\mathbb{P}[A] = \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{343}{365}.$$

   Now the event that at least two share a birthday is not- A and so

$$\begin{aligned}\mathbb{P}[\text{at least two share}] &= \mathbb{P}[\text{not} - A] \\ &= 1 - \mathbb{P}[A] \\ &= 1 - \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{343}{365} \\ &\approx 0.5073 = 50.73\%.\end{aligned}$$

8. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is maximised?

   Solution: Let $x \in [0, 10]$ and cut the wire into two pieces: of length $10 - x$ and x and make, respectively, a square and an equilateral triangle:



The area of the square is straightforward. To find the area of the triangle we could use $A(\Delta) = \frac{1}{2}ab\sin C$ but where does this formula come from? We have

$$\sin(60°) = \frac{h}{x/3}$$
$$\Rightarrow h = \sin(60°) \cdot \frac{x}{3} = \frac{x}{2\sqrt{3}}.$$

Now we can do $A(\Delta) = \frac{1}{2}bh = \frac{1}{2} \cdot \frac{x}{3} \cdot \frac{x}{2\sqrt{3}} = \frac{x^2}{12\sqrt{3}}$, and so the total area is given by

$$A(x) = \left(\frac{10 - x}{4}\right)^2 + \frac{x^2}{12\sqrt{3}}.$$

The temptation at this point is to use knowledge of calculus to look for 'turning points' of A by solving

$$\frac{dA}{dx} = 0,$$

this gives a point $x_0 = \frac{90}{9 + 4\sqrt{3}} \approx 5.650$ and then the maximum is given by $A(x_0)$... this is how we are trained. However if we pursue our training a little further we find that

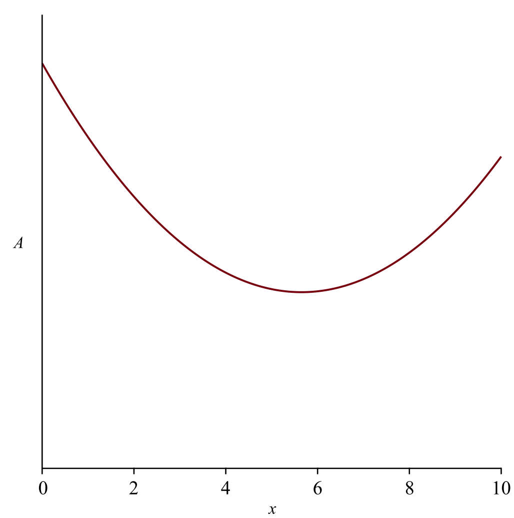
$$\frac{d^2A}{dx^2} = \frac{1}{8} + \frac{\sqrt{3}}{18} > 0,$$

which implies that x_0 gives a minimum rather than a max!

So where is the max? It is useful at this point to note that if multiplied out $A(x)$ is of the form

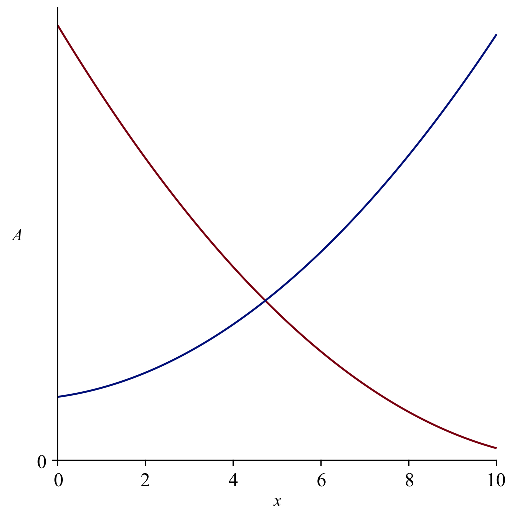
$$A(x) = ax^2 + bx + c, \qquad a > 0$$

that is a quadratic with '$\bigcup$'-geometry:



The geometry dictates that the maximum occurs at either of the end points.

As it happens the local minimum is between zero and ten but this need not be the case either: if it were not the area will be increasing or decreasing from $x = 0$ to $x = 10$ and so the maximum found at $x = 0$ or $x = 10$ anyway:



So, after all that discussion — and possibly making unnecessary calculations — the maximum is at $x = 0$ (all square) — or at $x = 10$ (all triangle). We calculate:

$$A(0) = 6.25 \text{ m}^2$$
$$A(10) = \frac{100}{12\sqrt{3}} \approx 4.811 \text{ m}^2.$$

Therefore the wire shouldn't be cut at all; and the wire turned into a square of side-length 2.5 m.

9. If every person in a group of 20 shook hands with all of the other people in the group, how many more handshakes take place than if the group were to split themselves into two groups of 10 and each person only shook hands with the other nine people in their group?

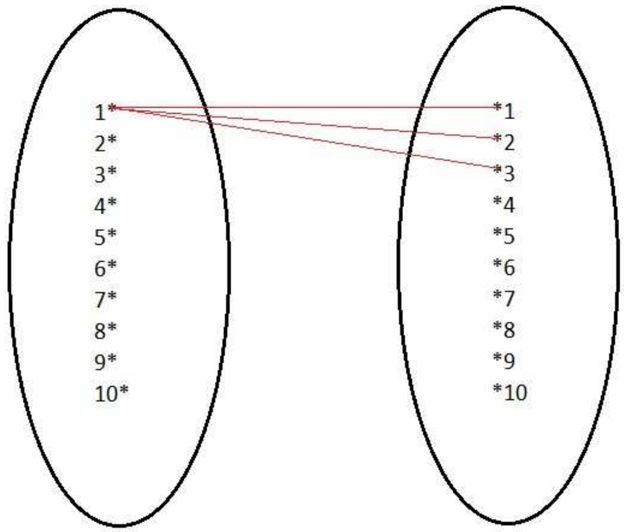
   Solution: There is a straightforward way of doing this: calculate the number of handshakes with 20 vs 2 groups of 10:

$$\binom{20}{2} = 190 \text{ vs } 90 = 2 \times \binom{10}{2},$$

   i.e, the answer is 100.

   It is possible to get the number 100 without calculating these numbers.

Imagine the group of 20 split into two groups of ten and suppose everyone in each group has shook the hands of everyone in their group. How many more handshakes will it take for everyone to shake the hand of everyone in the other group too?



Each person in the first group must shake hands with ten more people (in the second group). This gives $10 \times 10 = 100$ extra shakes to have everyone in the larger group of 20 having shook each other's hands.

10. There are eight identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. What is the minimum number of weighings needed to identify the fake coin with a two-pan balance scale without weights?

    Solution: Three is a reasonable answer (*binary search*) but it can actually be done with two.

    First separate the coins into subsets $S_1 = \{C_1, C_2, C_3\}$, $S_2 = \{C_4, C_5, C_6\}$, and $S_3 = \{C_7, C_8\}$. Now we weigh S_1 vs S_2. If these are equal the counterfeit coin is C_7 or C_8, and an additional weighing of C_7 vs C_8 will identify the counterfeit.

    If $S_1 \neq S_2$ (in weight), say $S_1 < S_2$, then the counterfeit coin is one of $\{C_1, C_2, C_3\}$. Weighing C_1 vs C_2 can go two ways. Either they are equal: in which case C_3 is the counterfeit, or one is lighter, say $C_1 < C_2$, in which case C_1 is the counterfeit.

We can show all eight coins are identifiable as the counterfeit using a tree diagram: